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Multi-variable Port Hamiltonian Model of Piezoelectric Material

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***Abstract*— In this paper, the dynamics of a piezoelectric material is presented within the new framework of multi-variable distributed port Hamiltonian systems. This class of infinite dimensional system is quite general, thus allowing the description of several physical phenomena, such as heat conduction, elasticity, electromagnetism and, of course, piezoelectricity. The key point is the generalization of the notion of finite dimensional Dirac structure in order to deal with an infinite dimensional space of power variables. In this way, the dynamics of the system results from the interconnection of a proper set of *elements*, each of them characterized by a particular energetic behavior, while the interaction with the environment is described in terms of mechanical and electrical boundary ports.**

I. INTRODUCTION

Piezoelectric materials have the ability to convert mechanical energy into electrical energy and, vice versa, the ability to convert electrical energy into mechanical energy. In particular, these materials produce an electrical charge under deformation, where the amount of charge depends on the deformation magnitude (*direct piezoelectric effect*), or, conversely, they become strained when an electric field is applied, with the magnitude of the developed strain depending on the field strength (*converse piezoelectric effect*), [1]. From the network modeling prospective [2], these are electro-mechanical system, that is the dynamics is the result of the interaction between the mechanical end the electrical domain. Interaction is simply power exchange via a power conserving network structure, mathematically described by Dirac structure, [3], [4], generalization of the well-known Kirchhoff laws of circuit theory.

Once the Dirac structure is defined, the dynamics of the system is specified when the space of energy (state) variables and the energy (Hamiltonian) function are given. The port Hamiltonian formalism [5], [6] is based on these ideas, thus allowing the description of a wide class of finite dimensional non-linear systems, such as mechanical, electro-mechanical, hydraulic and chemical ones. Recently, it has been extended in order to cope with the distributed parameter case by introducing the notion of infinite dimensional interconnection structure (Stokes–Dirac structure), [7]–[9].

In particular, in [10], a simple Stokes–Dirac structure has been presented as the starting point for the description in port Hamiltonian form of the telegrapher equation, of Maxwell's equations and of the vibrating string equation. In [8], this Stokes-Dirac structure has been modified in order to model fluid dynamical systems and in [11], [12] to model the Timoshenko beam equation. Moreover, in [9], the new class of multi-variable distributed parameter port Hamiltonian systems (mdpH systems) in which the interconnection, damping and input/output matrices of the finite dimensional case are replaced by (constant) matrix differential operators, is discussed.

In this paper, the mdpH formulation of the piezoelectric material dynamics in the linear case is presented. The main results concerns the definition of a Stokes–Dirac structure able to describe the *internal* and *external* interconnections, that is the way in which the electrical and mechanical domains interact within the system and the way in which the system can exchange power with the environment through the mechanical and electrical power ports defined on the boundary of the domain. Moreover, their relation with system dynamics, differently from [13], is suitable of an elegant interpretation in terms of network structure.

This paper is organized as follows. In Sect. II, the mdpH formulation of infinite dimensional system is briefly discussed by introducing the notion of infinite dimensional interconnection structure and presenting the corresponding class of distributed parameter systems. In Sect. III, a short background on the piezoelectric effect is given by presenting both the constitutive relations either the dynamical equations of the material. In Sect. IV, the dynamics of a piezoelectric material is presented within the mdpH framework by introducing the corresponding Stokes–Dirac structure and, then, in Sect. V, the dynamics of a one-dimensional piezoelectric bar (piezoelectric coupling) in port Hamiltonian form is deduced. In Sect. VI, the problem of interconnecting a piezoelectric actuator/sensor to a flexible structure is approached both in terms of composition of Dirac structure either by introducing a set of constraints (linear) on the energy variables. Finally, conclusion are discussed in Sect. VII.

II. PORT HAMILTONIAN FORMULATION OF MULTI-VARIABLE DISTRIBUTED SYSTEMS

A. Basics on Dirac structures

Consider an n-dimensional linear space $\mathcal{F}$ and denote by $\mathcal{E} \equiv \mathcal{F}^*$ its dual, that is the space of linear operator e :

$\mathcal{F} \to \mathbb{R}$. The elements belonging to $\mathcal{F}$ are called *flows* (e.g. velocities and currents), while the elements in $\mathcal{E}$ are called *efforts* (i.e. forces and voltages). Flows and efforts are the *port variables*, that is the input and output signals, whose combination gives the power flowing inside the physical system. The space $\mathcal{F}\times\mathcal{E}$ is called space of power variables.

Given an effort $e \in \mathcal{E}$ and a flow $f \in \mathcal{F}$, define the associated power P as $\langle e, f\rangle = e(f) \in \mathbb{R}$, where $\langle\cdot,\cdot\rangle$ is the *dual product* between f and e. Based on the dual product, the following linear operator is well-defined.

Definition 2.1 (+pairing operator): Consider the space of power variables $\mathcal{F}\times\mathcal{E}$. The following symmetric bilinear form is well-defined:

$$\ll (f_1, e_1), (f_2, e_2) \gg \; := \langle e_1, f_2\rangle + \langle e_2, f_1\rangle \qquad (1)$$

with $(f_i, e_i) \in \mathcal{F}\times\mathcal{E}$, $i = 1, 2$; $\ll\cdot,\cdot\gg$ is called +pairing operator.

Consider a linear subspace $\mathbb{S} \subset \mathcal{F}\times\mathcal{E}$ of dimension m and denote by $\mathbb{S}^\perp$ its orthogonal complement with respect to the +pairing operator (1), which is again a linear subspace of $\mathcal{F}\times\mathcal{E}$ with dimension $2n - m$ since (1) is a non-degenerate form. Based on the +pairing operator (1), it is possible to give the fundamental definition of Dirac structure, that is the basic mathematical tool that is used to describe the interconnection structure between physical systems.

Definition 2.2 (Dirac structure): Consider the space of power variables $\mathcal{F}\times\mathcal{E}$ and the symmetric bilinear form (1). A (constant) Dirac structure on $\mathcal{F}$ is a linear subspace $\mathbb{D} \subset \mathcal{F}\times\mathcal{E}$ such that

$$\mathbb{D} = \mathbb{D}^\perp$$

Note 2.1: It is possible to prove that the dimension of a Dirac structure $\mathbb{D}$ on an n-dimensional space $\mathcal{F}$ is equal to n. Moreover, suppose that $(f, e) \in \mathbb{D}$; from (1), we have that

$$0 = \ll (f, e), (f, e) \gg = 2\langle e, f\rangle$$

Then, it can be deduced that, for every $(f, e) \in \mathbb{D}$,

$$\langle e, f\rangle = 0$$

or, equivalently, that every Dirac structure $\mathbb{D}$ on $\mathcal{F}$ defines a power-conserving relation between power variables $(f, e) \in \mathcal{F}\times\mathcal{E}$.

B. Constant Stokes–Dirac structures

Denote by $\mathcal{Z}$ a compact subset of $\mathbb{R}^d$ representing the spatial domain of the distributed parameter system. Then, denote by $\mathcal{U}$ and $\mathcal{V}$ two sets of *smooth* functions from $\mathcal{Z}$ to $\mathbb{R}^{q_u}$ and $\mathbb{R}^{q_v}$ respectively.

Definition 2.3 (constant differential operator): A constant matrix differential operator of order N is a map L from $\mathcal{U}$ to $\mathcal{V}$ such that, given $u = (u^1, \dots, u^{q_u}) \in \mathcal{U}$ and $v = (v^1, \dots, v^{q_v}) \in \mathcal{V}$

$$v = Lu \iff v^b := \sum_{\#\alpha=0}^{N} P^\alpha_{a,b} D^\alpha u^a \qquad (2)$$

where $\alpha := \{\alpha_1, \dots, \alpha_d\}$ is a multi-index of order $\#\alpha := \sum_{i=1}^{d} \alpha_i$, P^α are a set of constant $q_u \times q_v$ matrices and $D^\alpha := \partial^{\alpha_1}_{z_1} \cdots \partial^{\alpha_d}_{z_d}$ is an operator resulting from a combination of spatial derivatives. Note that, in (2), the sum is intended over all the possible multi-indexes α with order 0 to N and, implicitly, on a from 1 to q.

Definition 2.4 (formal adjoint): Consider the constant matrix differential operator (2). Its formal adjoint is the map L^* from $\mathcal{V}$ to $\mathcal{U}$ such that

$$u = L^* v \iff u^b := \sum_{\#\alpha=0}^{N} (-1)^{\#\alpha} P^\alpha_{b,a} D^\alpha v^a \quad (3)$$

Definition 2.5 (skew-adjoint diff. op.): Denote by J a constant matrix differential operator. Then, J is *skew-adjoint* if and only if $J = -J^*$.

An important relation between a differential operator and its adjoint is expressed by the following lemma, which generalizes an analogous result presented in [14] to the multi variable case. This result is fundamental in the definition of Stokes–Dirac structure and, basically, it generalizes the well-known integration by parts formula.

Lemma 2.1: Consider a matrix differential operator L and denote by L^* its formal adjoint. Then, for every functions $u \in \mathcal{U}$ and $v \in \mathcal{V}$, we have that

$$\int_{\mathcal{Z}} \left[v^{\mathrm{T}} L u - u^{\mathrm{T}} L^* v\right] \mathrm{dV} = \int_{\partial\mathcal{Z}} B_L(u, v) \cdot \mathrm{dA} \qquad (4)$$

where B_L is a differential operator induced on $\partial\mathcal{Z}$ by L.

Note 2.2: Given $u \in \mathcal{U}$ and $v \in \mathcal{V}$, from the Stokes' theorem, it is well known that relation (4) can be equivalently written as

$$v^{\mathrm{T}} L u - u^{\mathrm{T}} L^* v = \operatorname{div} B_L(u, v)$$

that is $v^{\mathrm{T}} L u - u^{\mathrm{T}} L^* v$ can be expressed in divergence form. Moreover, it is important to note that B_L is a constant differential operator, that is the quantity $B_L(u, v)$ is a constant linear combination of the functions u and v together with their spatial derivatives up to a certain order and depending on L or, equivalently, a constant linear combination of $B_{\mathcal{Z}}(u)$ and $B_{\mathcal{Z}}(v)$, where $B_{\mathcal{Z}}$ is an operator providing a vector with all the spatial derivatives required in (21).

Corollary 2.2: Consider a skew-adjoint matrix differential operator J. Then, for every functions $u \in \mathcal{U}$ and $v \in \mathcal{V}$ with $q_u = q_v$, we have that

$$\int_{\mathcal{Z}} \left[v^{\mathrm{T}} J u + u^{\mathrm{T}} J v\right] \mathrm{dV} = \int_{\partial\mathcal{Z}} B_J(u, v) \cdot \mathrm{dA} \qquad (5)$$

where B_J is a non-degenerate symmetric differential operator on $\partial\mathcal{Z}$ depending on the differential operator J.

As in finite dimensions, the definition of a power conserving interconnection structure is possible once the notion of power is properly introduced. Denote by $\mathcal{F}$ the space of flows and assume that $\mathcal{F}$ is the space of *smooth* functions from the compact set $\mathcal{Z} \subset \mathbb{R}^d$ to $\mathbb{R}^q$. As far as concerns the space of efforts $\mathcal{E}$, assume for simplicity that $\mathcal{E} \equiv \mathcal{F}$. Then, given $f = (f^1, \dots, f^q) \in \mathcal{F}$ and $e = (e^1, \dots, e^q) \in \mathcal{E}$, define the dual product as follows:

$$\langle e, f\rangle := \int_{\mathcal{Z}} \sum_{i=1}^{q} e^i f^i \,\mathrm{dV} = \int_{\mathcal{Z}} e^{\mathrm{T}} f \,\mathrm{dV}$$

From Def. 2.1, the +pairing operator on $\mathcal{F} \times \mathcal{E}$ is given by

$$\ll (f_1, e_1), (f_2, e_2) \gg := \int_{\mathcal{Z}} \left[e_1^{\mathrm{T}} f_2 + e_2^{\mathrm{T}} f_1\right] \,\mathrm{dV}$$

with $(f_1, e_1), (f_2, e_2) \in \mathcal{F} \times \mathcal{E}$. A general class of constant Stokes–Dirac structures is provided by the following theorem, [9].

Theorem 2.3 (constant Stokes–Dirac structure): Denote by $\mathcal{Z} \subset \mathbb{R}^d$ a compact set and by $\mathcal{F} = (\mathcal{F}_s, \mathcal{F}_r, \mathcal{F}_d)$ a space of vector values smooth functions on $\mathcal{Z}$, the space of flows. For simplicity, suppose that $\mathcal{E} = (\mathcal{E}_s, \mathcal{E}_r, \mathcal{E}_d) \equiv \mathcal{F}$ is the space of efforts. Moreover, assume that J, G_r and G_d are constant matrix differential operator such that $J : \mathcal{E}_s \to \mathcal{F}_s$ and $J = -J^*$, $G_r : \mathcal{F}_r \to \mathcal{F}_s$ and $G_d : \mathcal{F}_d \to \mathcal{F}_s$. Then,

$$\begin{aligned} \mathbb{D} := \{\, (f, e, w) \in \mathcal{F} \times \mathcal{E} \times \mathcal{W} \;|\; & \\ f_s = -J e_s - G_r f_r - G_d f_d & \\ e_r = G_r^* e_s, \quad e_d = G_d^* e_s & \\ w = B_{\mathcal{Z}}(e_s, f_r, f_d) \,\} & \end{aligned} \tag{6}$$

is a Stokes–Dirac structure with respect to the pairing

$$\begin{aligned} &\ll (f_1, e_1, w_1), (f_2, e_2, w_2) \gg_{\{J, G_r, G_d\}} := \\ &:= \int_{\mathcal{Z}} \left[e_1^{\mathrm{T}} f_2 + e_2^{\mathrm{T}} f_1\right] \,\mathrm{dV} + \int_{\partial\mathcal{Z}} B(w_1, w_2) \cdot \mathrm{dA} \end{aligned} \tag{7}$$

where $B_{\mathcal{Z}}$ is the analogous of the boundary operator of Note 2.2 and $B(\cdot, \cdot)$ is the boundary differential operator induced by J, G_r and G_d on $\partial\mathcal{Z}$.

Note 2.3: Suppose that $(f, e, w) \in \mathbb{D}$. From (7), we have that

$$\int_{\mathcal{Z}} \left[e_s^{\mathrm{T}} f_s + e_r^{\mathrm{T}} f_r + e_d^{\mathrm{T}} f_d\right] \,\mathrm{dV} + \frac{1}{2} \int_{\partial\mathcal{Z}} B(w_1, w_2) \cdot \mathrm{dA} = 0$$

or, equivalently, that

$$\begin{aligned} -\int_{\mathcal{Z}} e_s^{\mathrm{T}} f_s = \int_{\mathcal{Z}} e_r^{\mathrm{T}} f_r \,\mathrm{dV} + \int_{\mathcal{Z}} e_d^{\mathrm{T}} f_d \,\mathrm{dV} \\ + \frac{1}{2} \int_{\partial\mathcal{Z}} B(w_1, w_2) \cdot \mathrm{dA} \end{aligned} \tag{8}$$

This relation, which is a direct consequence of the definition of Dirac structure, expresses the property that the variation of internal energy is equal to the sum of the dissipated power with the power provided to the system through the domain $\mathcal{Z}$ and the boundary $\partial\mathcal{Z}$.

C. Infinite dimensional port Hamiltonian systems

As in finite dimensions, the dynamics of a distributed parameter system can be obtained from its Stokes–Dirac structure once the power ports are terminated on the corresponding elements, that is the input/output behavior of the *components* are specified.

Denote by $\mathcal{X}$ the space of smooth real valued functions on $[0, +\infty) \times \mathcal{Z}$ representing the space of energy *configuration.* The total energy is a functional $\mathcal{H} : \mathcal{X} \to \mathbb{R}$ such that

$$\mathcal{H}(x) = \int_{\mathcal{Z}} H(z, x) \,\mathrm{dV}$$

where H is the energy density. As proposed in [10], the port behavior of the energy storing element is given by

$$f_s = -\frac{\partial x}{\partial t} \qquad e_s = \delta_x \mathcal{H} \tag{9}$$

where $\delta_x \mathcal{H}$ is the variational derivative of the Hamiltonian with respect to the energy configuration. Linear dissipation can be introduced by imposing that

$$f_r = -Y_r e_r, \quad \text{with} \quad \int_{\mathcal{Z}} e_r^{\mathrm{T}} Y_r e_r \,\mathrm{dV} \geq 0 \tag{10}$$

where $Y_r : \mathcal{E}_r \to \mathcal{F}_r$ is a linear operator. If $\tilde{B}_{\mathcal{Z}}$ is the boundary operator introduced in (6), from (10) we have that

$$\begin{aligned} \tilde{B}_{\mathcal{Z}}(e_s, f_r, f_d) &= \tilde{B}_{\mathcal{Z}}(e_s, -Y_r G_r^* e_s, f_d) \\ &=: B_{\mathcal{Z}}(e_s, f_d) \end{aligned} \tag{11}$$

and then the boundary terms can be computed as $w = B_{\mathcal{Z}}(e_s, f_d)$. Consequently, taking into account (6), (9), (10) and (11), the following definition makes sense.

Definition 2.6 (mdpH system): Denote by $\mathcal{X}$ the space of vector value smooth functions on $[0, +\infty) \times \mathcal{Z}$ (energy configurations), by $\mathcal{F}_d$ the space of vector value smooth functions on $\mathcal{Z}$ (distributed flows) and assume that $\mathcal{E}_d \equiv \mathcal{F}_d$ is its dual (distributed efforts) and by $\mathcal{W}$ the space of vector value smooth functions on $\partial\mathcal{Z}$ representing the boundary terms. Moreover, denote by J a skew-adjoint differential operator, by G_d a differential operator and by $B_{\mathcal{Z}}$ the boundary operator defined in (11).

If $\mathcal{H} : \mathcal{X} \to \mathbb{R}$ is the Hamiltonian function, the general formulation of a multi-variable distributed port Hamiltonian system with constant Stokes–Dirac structure is

$$\left\{ \begin{array}{rcl} \dfrac{\partial x}{\partial t} &=& (J - R)\, \delta_x \mathcal{H} + G_d f_d \\ e_d &=& G_d^* \delta_x \mathcal{H} \\ w &=& B_{\mathcal{Z}}(\delta_x \mathcal{H}, f_d) \end{array} \right. \tag{12}$$

where $R := G_r Y_r G_r^*$ is a differential operator taking into account energy dissipation and $(f_d, e_d) \in \mathcal{F}_d \times \mathcal{E}_d$.

Proposition 2.4: Consider the mdpH system (12). Then, the following energy balance inequality holds:

$$\begin{aligned} \frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} &= -\int_{\mathcal{Z}} (\delta_x \mathcal{H})^{\mathrm{T}} R\, \delta_x \mathcal{H} \,\mathrm{dV} + \int_{\mathcal{Z}} e_d^{\mathrm{T}} f_d \,\mathrm{dV} \\ &\quad + \frac{1}{2} \int_{\partial\mathcal{Z}} B(w_1, w_2) \cdot \mathrm{dA} \\ &\leq \int_{\mathcal{Z}} e_d^{\mathrm{T}} f_d \,\mathrm{dV} + \frac{1}{2} \int_{\partial\mathcal{Z}} B(w_1, w_2) \cdot \mathrm{dA} \end{aligned} \tag{13}$$

Proof: From (9), we have that

$$-\int_{\mathcal{Z}} e_s^{\mathrm{T}} f_s \,\mathrm{dV} = \int_{\mathcal{Z}} (\delta_x \mathcal{H})^{\mathrm{T}} \frac{\partial x}{\partial t} \,\mathrm{dV} = \frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t}$$

Then, (13) is immediate from (8) and (10). ■

Note 2.4: Relation (13) expresses an obvious property of physical systems, that is the variation of internal energy is less or equal (if no dissipation is present) to the power provided to the system. In the case of distributed parameter system, the power can flow inside the system either through the boundary and/or the spatial domain.

III. Background on the piezoelectric effect

The dynamics of the piezoelectric material presented in this section is valid under the assumptions of linear behavior, no thermal effects and quasi static electric field, [1]. The linear material behavior relating stress σ and strain ϵ in an elastic body is described in coordinates by the *constitutive relation* (Hooke's law) $\epsilon_{ij} = S_{ijkl}\sigma_{kl}$, where ϵ_{ij} and σ_{kl} are the components of the strain and stress tensors while S_{ijkl} are the components of the compliance tensor. Otherwise, the linear constitutive relation of a dielectric medium is described by $D_i = \varepsilon_{ij}E_j$, where D_i is the electric flux density, E_j is the electric field vector and ε_{ij} is the electric permittivity tensor. For a piezoelectric material, the mechanical and electrical constitutive relations are coupled:

$$\begin{aligned} \epsilon_{ij} &= S^E_{ijkl}\sigma_{kl} + d_{kij}E_k \\ D_i &= d_{ikl}\sigma_{kl} + \varepsilon^\sigma_{ik}E_k \end{aligned} \tag{14}$$

where d_{ijk} is the piezoelectric charge tensor which measures the amount of strain developed by an electric field in the absence of stress or, conversely, the amount of electric flux density due to a stress in a zero electric field. Moreover, S^E is the compliance tensor for a constant electric field and ε^σ is the permittivity tensor under when the stress is constant. In matrix notation, (14) can be written as

$$\begin{bmatrix} \epsilon \\ D \end{bmatrix} = \begin{bmatrix} S^E & d^{\mathrm{T}} \\ d & \varepsilon^\sigma \end{bmatrix} \begin{bmatrix} \sigma \\ E \end{bmatrix}$$

The strain and the electric field tensors are related to the components w_i of the displacement and to the electric potential field ϕ via the compatibility equations:

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i}\right) \qquad E_i = -\frac{\partial \phi}{\partial x_i}$$

The dynamical equilibrium of a continuous piezoelectric media is described by means of two equations. The first one is the Newton's law describing the balance of mechanical forces, while the second one describes the balance of electrical charges. These equations of motion are respectively given by

$$\sum_j \frac{\partial \sigma_{ij}}{\partial x_j} + f_i = \rho \ddot{w}_i + c_d \dot{w}_i \tag{15a}$$

$$\sum_i \frac{\partial D_i}{\partial x_i} = q \tag{15b}$$

where f_i are the components of the boy force vector, ρ is the mass density, c_d the viscous damping factor and q the free charge density. If written in integral form, (15b) states that the electric flux emanating from a closed surface is equal to the total charge enclosed by the surface.

IV. mdpH model of piezoelectric material

Denote by $\mathcal{Z} \subset \mathbb{R}^3$ the spatial domain and assume that ϵ, $p = \rho\dot{w}$ (mechanical momentum) and D are the energy variables and that σ, $\dot{w}$ and E the corresponding co-energy variables. Consequently, the total energy is given by

$$\mathcal{H}(\epsilon, p, D) = \frac{1}{2}\int_{\mathcal{Z}} \left(\frac{\|p\|^2}{\rho} + \sigma^{\mathrm{T}}\epsilon + E^{\mathrm{T}}D\right) \mathrm{dV} \tag{16}$$

In order to write the model of piezoelectric material in mdpH form, it is necessary to identify its Stokes–Dirac structure. We give the following corollary to Theorem 2.3:

Corollary 4.1: Denote by $\mathcal{F} = \mathcal{F}_s \times \mathcal{F}_c \times \mathcal{F}_q$ a space of vector value smooth functions on $\mathcal{Z}$ (space of flows) and assume that $\mathcal{E} = \mathcal{E}_s \times \mathcal{E}_c \times \mathcal{E}_\phi \equiv \mathcal{F}$ is its dual space (space of efforts), with $\mathcal{E}_\phi \equiv \mathcal{F}_q$. Moreover, denote by J and G_c a couple of multi-variable differential operators, with $J = -J^*$. Then, the following set:

$$\begin{aligned} \mathbb{D}_{pe} := \{ (f, e, w) \in \mathcal{F} \times \mathcal{E} \times \mathcal{W} \mid \; & f_s = -Je_s \\ & f_q = -G_c f_c, \;\; G_c^* e_\phi = e_c \\ & w = B_{\mathcal{Z}}(e_s, f_c, e_c) \} \end{aligned} \tag{17}$$

is a Stokes–Dirac structure on $\mathcal{F}$ with respect to the pairing

$$\begin{aligned} &\ll (f_1, e_1, w_1), (f_2, e_2, w_2) \gg_{pe} = \\ &= \int_{\mathcal{Z}} \left(e_1^{\mathrm{T}} f_2 + e_2^{\mathrm{T}} f_1\right) \mathrm{dV} + \int_{\partial\mathcal{Z}} B(w_1, w_2) \cdot \mathrm{dA} \end{aligned}$$

where $\mathcal{W} = \{w \mid w = B_{\mathcal{Z}}(e_s, f_c, e_c), \; \forall (e_s, f_c, e_c) \in \mathcal{E}_s \times \mathcal{F}_c \times \mathcal{E}_c\}$ is the space of boundary variables and $B_{\mathcal{Z}}$ and $B(\cdot,\cdot)$ are the boundary operator and the differential operator on $\partial\mathcal{Z}$ induced by J and G_c respectively.

Proof: From Theorem 2.3, both the relations $f_s = -Je_s$ and $f_q = -G_c f_c$, $G_c^* e_\phi = e_c$ define a Stokes–Dirac structure once the proper set of boundary variables and the corresponding pairing is determined. Clearly, (17) results from the Cartesian product of these Stokes–Dirac structure and, then, it is a Stokes–Dirac structure, [4]. ■

As regard the piezoelectric material, assume that

$$\begin{aligned} f_s &= (f_{\epsilon_{11}}, \dots, f_{\epsilon_{12}}, f_{p_1}, \dots, f_{p_3}) \\ e_s &= (e_{\epsilon_{11}}, \dots, e_{\epsilon_{12}}, e_{p_1}, \dots, e_{p_3}) \\ f_c &= (f_{D_1}, \dots, f_{D_3}) \\ e_c &= (e_{D_1}, \dots, e_{D_3}) \end{aligned}$$

and that $\mathcal{F}_s = \mathcal{E}_s \subset (L^2(\mathcal{Z}))^9$, $\mathcal{F}_c = \mathcal{E}_c \subset (L^2(\mathcal{Z}))^3$ and $\mathcal{F}_q = \mathcal{E}_\phi \subset L^2(\mathcal{Z})$. Moreover, in (17), define J and G_c as follows:

$$J = \begin{bmatrix} 0 & \tilde{J} \\ -\tilde{J}^* & 0 \end{bmatrix}, \text{ with } \tilde{J} = \begin{bmatrix} \partial_{x_1} & 0 & 0 \\ 0 & \partial_{x_2} & 0 \\ 0 & 0 & \partial_{x_3} \\ 0 & \partial_{x_3} & \partial_{x_2} \\ \partial_{x_3} & 0 & \partial_{x_1} \\ \partial_{x_2} & \partial_{x_1} & 0 \end{bmatrix} \tag{18}$$

$$G_c = \begin{bmatrix} \partial_{x_1} & \partial_{x_2} & \partial_{x_3} \end{bmatrix}$$

Clearly, $J = -J^*$. In order to compute $B_{\mathcal{Z}}$ and $B(\cdot,\cdot)$, consider $(f, e) \in \mathcal{F} \times \mathcal{E}$ and re-write the associated power flow in divergence form, that is as integral over the boundary of the domain of a certain differential form:

$$\int_Z e^{\mathrm{T}} f \,\mathrm{dV} = -\int_{\partial Z} [\, e_{p_1}\, e_{p_2}\, e_{p_3}] \begin{bmatrix} e_{\epsilon_{11}} & e_{\epsilon_{11}} & e_{\epsilon_{11}} \\ e_{\epsilon_{11}} & e_{\epsilon_{11}} & e_{\epsilon_{11}} \\ e_{\epsilon_{11}} & e_{\epsilon_{11}} & e_{\epsilon_{11}} \end{bmatrix} \cdot \mathrm{dA} - \int_{\partial Z} e_\phi \, [\, f_{D_1}\, f_{D_2}\, f_{D_3}] \cdot \mathrm{dA}$$

Consequently, $w = \left(e_p|_{\partial Z}\,,\, e_\epsilon|_{\partial Z}\,,\, e_\phi|_{\partial Z}\,,\, f_D|_{\partial Z}\right)$ and

$$\frac{1}{2}B(w,w) = [\, e_{p_1}\, e_{p_2}\, e_{p_3}] \begin{bmatrix} e_{\epsilon_{11}} & e_{\epsilon_{11}} & e_{\epsilon_{11}} \\ e_{\epsilon_{11}} & e_{\epsilon_{11}} & e_{\epsilon_{11}} \\ e_{\epsilon_{11}} & e_{\epsilon_{11}} & e_{\epsilon_{11}} \end{bmatrix} - e_\phi \, [\, f_{D_1}\, f_{D_2}\, f_{D_3}]$$

Note that, if $f_{b,M} = e_p|_{\partial Z}$, $e_{b,M} = e_\epsilon|_{\partial Z}$, $f_{b,E} = f_D|_{\partial Z}$ and $e_{b,E} = e_\phi|_{\mathrm{d}Z}$, then

$$\frac{1}{2}B(w,w) = f_{b,M}^{\mathrm{T}} e_{b,M} + f_{b,E}^{\mathrm{T}} e_{b,E}$$

that is on the boundary it is possible to define a couple of (distributed) power ports, the mechanical (M) and the electrical (E) one, and the associated power flows results to result from the dual product of the corresponding flows and efforts.

Given the Hamiltonian function (16), the dynamics of the system can be specified as follows:

$$\begin{aligned} &f_\epsilon = -\dot\epsilon \quad f_p = -\dot p \quad f_D = -\dot D \quad f_q = -\dot q \\ &e_\epsilon = \delta_\epsilon \mathcal{H} \quad e_p = \delta_p \mathcal{H} \quad e_D = \delta_D \mathcal{H} \quad e_\phi = \phi \end{aligned} \tag{19}$$

where $\dot q$ and ϕ are external signals.

V. Case study: the piezoelectric coupling

Consider a one-dimensional piezoelectric bar of length L, that is $\mathcal{Z} = [0, L]$. Under the hypothesis that the free charge density is equal to zero ($q = 0$) and that there are no external forces and no damping effects acting on the system ($f = 0$ and $c_d = 0$ in (15a)), it is possible to deduce the one dimensional model for the longitudinal vibration of the bar, [1].

The constitutive relation (14) becomes

$$\epsilon = S^E \sigma + dE \qquad D = d\sigma + \varepsilon^\sigma E$$

and the Hamiltonian (16)

$$\mathcal{H}(\epsilon, p, D) = \frac{1}{2}\int_0^L \left(p^2/\rho + \epsilon\sigma + DE\right) \,\mathrm{dz}$$

From (17), (18) and (19), the equations of motions in mdpH form can be written as follows:

$$\begin{cases} \dot\epsilon = \partial_z \delta_p \mathcal{H} \;\; (= \partial_z(p/\rho)) \\ \dot p = \partial_z \delta_\epsilon \mathcal{H} \;\; (= \partial_z \sigma) \\ 0 = \partial_z \dot D \\ E = \partial_z \phi \end{cases} \tag{20}$$

These equations can be properly arranged in order to obtain a couple of PDEs expressing the relation between the displacement w and the potential ϕ. If $E_{pe} = 1/S^E$ and $k = d/\sqrt{E_{pe}/\varepsilon^\sigma}$ are the Young's modulus in the z-directions and the electro-mechanical coupling factor respectively, the constitutive relation can be written as

$$\sigma = E_{pe}\epsilon - dE_{pe}E \qquad D = dE_{pe}\epsilon + \varepsilon^\sigma(1 - k^2)E$$

Since $\epsilon = \partial_z w$ and $p = \rho w$, from the second and fourth equations in (20) we obtain that

$$\rho\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial z}\left(E_{pe}\epsilon - dE_{pe}E\right) = E_{pe}\left(\frac{\partial^2 w}{\partial z^2} + d\frac{\partial^2 \phi}{\partial z^2}\right) \tag{21a}$$

while, from the third and fourth relations in (20)

$$0 = \frac{\partial D}{\partial z} = dE_{pe}\frac{\partial \epsilon}{\partial z} + \varepsilon^\sigma(1 - k^2)\frac{\partial E}{\partial z}$$

that is

$$\frac{\partial^2 \phi}{\partial z^2} = \frac{k^2}{d(1 - k^2)}\frac{\partial^2 w}{\partial z^2} \tag{21b}$$

The couple of relations (21) describe the longitudinal vibration of the piezoelectric bar under the effect of an electric potential ϕ.

VI. Interconnection of a piezoelectric actuator with a flexible structure

Due to the converse piezoelectric effect, under the effect of a potential field the piezoelectric material can generate a strain force which is proportional to the strength of the electric field. This property can be fruitfully used in order to damp vibrations over a flexible beam or plate or to implement an active control of acoustic systems. In oder to act over a flexible structure by means of a piezoelectric actuator, it is necessary to interconnect both the system. In this section, this procedure is described within the framework of mdpH systems.

Consider the Stokes–Dirac structure of the piezoelectric material (17), here rewritten for simplicity:

$$\begin{cases} f_{\epsilon,pe} = -\tilde J e_{p,pe} \\ f_{p,pe} = \tilde J^* e_{\epsilon,pe} \\ \dot q = -\operatorname{div} f_D \\ e_D = -\operatorname{grad}\phi \end{cases}$$

where $\tilde J$ is the differential operator introduced in (18) which models the linear behavior of the mechanical structure. The boundary variables are $e_{p,pe}$ and $e_{\epsilon,pe}$, for the mechanical part, and ϕ and f_D for the electrical one. Moreover denote by $\mathcal{Z}_{pe}$ the spatial domain of the piezoelectric material. Suppose that the flexible structure behaves linearly and denote by $\mathcal{Z}_b$ its spatial domain. Then, the corresponding Stokes–Dirac structure is defined by the following relation:

$$\begin{cases} f_{\epsilon,b} = -\tilde J e_{p,b} \\ f_{p,b} = \tilde J^* e_{\epsilon,b} \end{cases}$$

where $e_{p,b}$ and $e_{\epsilon,b}$ are the boundary variables.

The piezoelectric actuator and the flexible structure can be interconnected if there is an intersection $\mathcal{Z}_{pe,b} := \partial\mathcal{Z}_{pe} \cap \partial\mathcal{Z}_b$ between the boundaries of their spatial domain. The interconnection between the two subsystems is power conserving if and only if $e_{p,pe} = e_{p,b}$ and $e_{\epsilon,pe} = -e_{\epsilon,b}$ on

$\mathcal{Z}_{pe,b}$. This relation imposes the continuity of speed at the interconnection and that the exchanged forces are equal in intensity, but with opposite direction, along the common portion of the boundary. Note that the continuity in velocity implies continuity of strain.

The Hamiltonian functions of the piezoelectric actuator and of the flexible structure are respectively given by

$$\mathcal{H}_{pe}(p_{pe}, \epsilon_{pe}, D) = \frac{1}{2}\int_{\mathcal{Z}_{pe}} \left(\frac{\|p_{pe}\|^2}{\rho} + \sigma_{pe}^{\mathrm{T}} \epsilon_{pe} + E^{\mathrm{T}} D \right) \mathrm{dV}$$
$$\mathcal{H}_b(p_b, \epsilon_b) = \frac{1}{2}\int_{\mathcal{Z}_b} \left(\frac{\|p_b\|^2}{\rho} + \sigma_b^{\mathrm{T}} \epsilon_b \right) \mathrm{dV}$$

and the total energy function of the resulting interconnected system will be $\mathcal{H}_{cl} = \mathcal{H}_{pe} + \mathcal{H}_b$. If the final system is autonomous, that is no force is acting on the $\partial\mathcal{Z}_{pe} \cup \partial\mathcal{Z}_b$, it is possible to write the following energy balance relation:

$$\frac{\mathrm{d}\mathcal{H}_{cl}}{\mathrm{d}t} = -\int_{\partial\mathcal{Z}_{pe} \setminus \mathcal{Z}_{pe,b}} \phi f_D \cdot \mathrm{dA}$$

where $-f_D \cdot \mathrm{dA}$ is the current flux through the boundary, denoted by j. If a measure of j is available, in order to stabilize the system it is possible to assume that $\phi = -\alpha j$, with $\alpha \geq 0$, since, in this way, $\dot{\mathcal{H}}_{cl} \leq 0$.

The problem of interconnecting a piezoelectric actuator to a flexible structure can be also tackled in a different way by supposing that $\mathcal{Z}_{pe} \subset \mathcal{Z}_b$ and, then, by imposing a speed constrain on $\mathcal{Z}_{pe}$. Before continuing, the following Corollary to Theorem 2.3 is necessary:

Corollary 6.1: Denote by $\mathcal{F} \times \mathcal{E}$ an infinite dimensional space of power variables, by J a skew-adjoint differential operator and by G a full-rank linear operator. A set of (independent) constraints on e can be introduced by imposing that $Ge = 0$. Then, the following set is a Stokes–Dirac structure on $\mathcal{F}$ with constraints on the effort variables:

$$\begin{aligned} \mathbb{D}_L = \{ \ (f,e) \in \mathcal{F} \times \mathcal{E} \ \mid \ & f = -Je + G^{\mathrm{T}}\lambda \\ & 0 = Ge, \quad w = B_J(e) \ \} \end{aligned}$$

Here, the vector value function λ plays the same role of the Lagrange multipliers in finite dimensions.

Consider the following function $\Psi : \mathcal{Z}_b \to \{0,1\}$ such that

$$\Psi(\xi) = \begin{cases} 1 & \text{if} \ \ \xi \in \mathcal{Z}_{pe} \\ 0 & \text{if} \ \ \xi \in \mathcal{Z}_b \end{cases}$$

Then, the velocity constrain $e_{p,pe} - e_{p,b} = 0$ on $\mathcal{Z}_{pe}$ can be easily introduced and the resulting Stokes–Dirac structure is given by

$$\begin{bmatrix} f_{\epsilon,b} \\ f_{p,b} \\ f_{\epsilon,pe} \\ f_{p,pe} \end{bmatrix} = -\begin{bmatrix} 0 & \tilde{J} & 0 & 0 \\ -\tilde{J}^* & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{J} \\ 0 & 0 & -\tilde{J}^* & 0 \end{bmatrix} \begin{bmatrix} e_{\epsilon,b} \\ e_{p,b} \\ e_{\epsilon,pe} \\ e_{p,pe} \end{bmatrix} + \Psi \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \lambda$$
$$0 = \Psi \begin{bmatrix} 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} e_{\epsilon,b} \\ e_{p,b} \\ e_{\epsilon,pe} \\ e_{p,pe} \end{bmatrix}$$
$$\dot{q} = \operatorname{div} f_D$$
$$e_D = \operatorname{grad} \phi$$

Since $e_{p,pe} = e_{p,b}$ on $\mathcal{Z}_{pe}$, also the continuity on strain is imposed.

VII. Conclusion

In this paper, the dynamics of a piezoelectric material is presented within the new framework of multi-variable distributed port Hamiltonian systems by introducing the corresponding Stokes–Dirac structure. As an example, the model of a one-dimensional piezoelectric bar (piezoelectric coupling) in port Hamiltonian form is discussed. Moreover, the problem of interconnecting a piezoelectric actuator/sensor to a flexible structure is approached both in terms of composition of Dirac structure either by introducing a set of constraints (linear) on the energy variables.

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